JOURNAL OF APPROXIMATION THEORY 64, 356-358 (1991)

Notes

Banach Spaces Not Antiproximinal in Their Second Dual

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Communicated by Frank Deutsch

Received February 13, 1989; revised July 31, 1990

We prove that $(l^1, |\cdot|)$ is not antiproximinal in $(l^1, |\cdot|)^{**}$, where $|\cdot|$ is the norm constructed in [1]. This fact shows that Davidson's equivalent norm fails to deliver on his promise. © 1991 Academic Press, Inc.

A subspace M is called antiproximinal in a Banach space X if the only vectors with closest approximants from M are the elements of M. A Banach space X is said to have the projection approximation property (PAP) if there is an increasing sequence (P_n) of commuting, finite rank idempotents in $\mathscr{B}(X)$ tending strongly to the identity operator. The consideration of whether X is antiproximinal in X^{**} was studied by Davidson [1], where it was claimed that if X has the PAP, then X has an equivalent norm $|\cdot|$ such that $(X, |\cdot|)$ is antiproximinal in $(X, |\cdot|)^{**}$. However, in this paper we prove that $(l^1, |\cdot|)$ is not antiproximinal in $(l^1, |\cdot|)^{**}$, where $|\cdot|$ is the norm constructed in [1]. This fact shows that Davidson's equivalent norm fails to deliver on his promise.

Let $(X, \|\cdot\|)$ be a Banach space with the PAP, (P_n) be an increasing sequence of commuting, finite rank idempotents in $\mathscr{B}(X)$ tending strongly to the identity operator, and $\|P_n\| = 1$, $\|I - P_n\| \leqslant 1$ for all n. Now let

$$Y = X \hat{\otimes} l^1 = \left\{(x_n) : x_n \in X, \sum \|x_n\| < \infty\right\}.$$

Given $\varepsilon > 0$, define a compact operator T from X into Y by

$$Tx = (2^{-n}\varepsilon P_n x).$$

In [1], Davidson constructed a new norm on X by

$$|x| = \|x\| + \|Tx\|, \qquad x \in X.$$

0021-9045/91 $3.00

To prove that $(X, |\cdot|)$ is antiproximinal in $(X, |\cdot|)^{**}$, the author used the assumption that T^{**} is injective. In fact, T^{**} need not be injective. We can prove that if the dual space X^* of X is non-separable, and T is any compact operator from X into Y, then T^{**} is not injective. To see this, we note that T is a compact operator, hence so is T^*. If T^{**} were injective, we could apply Theorem IV.8.4(c) [3, p. 232] to conclude that $\overline{\mathscr{R}(T^*)} = X^*$. Therefore X^* is separable, a contradiction.

It is well-known that if X has a Schauder basis, then X has the PAP, and the basic projections P_n $(n = 1, 2, ...)$ are increasing, commuting, idempotent, finite-rank operators. If X is l^p, $1 \leqslant p < \infty$, it is clear that $\|P_n\| = 1$ and $\|I - P_n\| = 1$, for all n.

EXAMPLE. Let $X = l^1$, $Y = X \hat{\otimes} l^1 = \{(x_n) : x_n \in X, \sum \|x_n\| < \infty\}$. $P_n x = \sum_{k=1}^n \xi_k e_k$, $x = (\xi_k) \in X$, $n = 1, 2, ...,$ and let (e_k) be the usual unit vector basis of l^1. Assume that the operator T and the norm $|\cdot|$ are as above. We claim that $(X, |\cdot|)$ is not antiproximinal in $(X, |\cdot|)^{**}$. To see this let y^* be an arbitrary element in Y^*. For each $e_n \in X$, we have

$$\begin{aligned} |\langle e_n, T^*y^* \rangle| &= |\langle Te_n, y^* \rangle| \\ &= |\langle (2^{-1}\varepsilon P_1 e_n, ..., 2^{-k}\varepsilon P_k e_n, ...), y^* \rangle| \\ &= |\langle (0, ..., 0, 2^{-n}\varepsilon e_n, 2^{-n-1}\varepsilon e_n, ...), y^* \rangle| \\ &\leqslant 2^{-n+1}\varepsilon \|y^*\| \to 0, \qquad n \to \infty, \end{aligned}$$

so $T^*y^* \in c_0$, consequently $T^*Y^* \subset c_0$. It is known [2] that $(l^1)^{**} = (l^1 \oplus (c_0)^0)$, where $(c_0)^0$ refers to the annihilator of c_0 in l_∞^* when c_0 is considered as a subspace of l_∞. Take $x^{**} \in X^{**}$, $x^{**} = (0, u)$, $u \in (c_0)^0$, $u \neq 0$. Then

$$\langle y^*, T^{**}x^{**} \rangle = \langle T^*y^*, x^{**} \rangle = 0 \qquad \text{for all } y^* \in Y^*.$$

It follows that $T^{**}x^{**} = 0$. Moreover,

$$\begin{aligned} d(x^{**}, X) := \inf_{x \in X} \|x - x^{**}\| &= \inf_{x \in X} \|(x, -u)\|_1 \\ &= \inf_{x \in X} (\|x\| + \|u\|) = \|u\| = \|x^{**}\|. \end{aligned}$$

By Lemma 2.5 [1, p. 206], we obtain

$$|x^{**}| = \|x^{**}\| + \|T^{**}x^{**}\| = \|x^{**}\|.$$

Thus $d(x^{**}, X) = \|x^{**}\| = |x^{**}|$. Also,

$$d'(x^{**}, X) := \inf_{x \in X} |x - x^{**}| \leqslant |x^{**}|,$$

hence

$$d'(x^{**}, X) = |x^{**}|.$$

This shows that x^{**} has a closest approximant in X with respect to the norm $|\cdot|$, and $x^{**} \notin X$. Thus $(X, |\cdot|)$ is not antiproximinal in $(X, |\cdot|)^{**}$.

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